<u>Reider's Theorem</u> ('88): X a smooth projective surface, L a nef line bundle on X.

(i.) Assume $L^2 \geq 5$ and that $x \in Bs|K_X + L|$. Then there is an effective divisor $D \subset X$ passing through x such that either

$D \cdot L = 0$ and $D^2 = -1$ or

$D \cdot L = 1$ and $D^2 = 0$

(ii.) Assume $L^2 \geq 10$ and Z a length two subscheme which fails to be separated by $|K_X + L|$, then there is an effective divisor $D \subset X$ through Z such that either

$D \cdot L = 0$ and $D^2 = -1$ or -2; or

$D \cdot L = 1$ and $D^2 = 0$ or -1; or

$D \cdot L = 2$ and $D^2 = 0$.

<u>Example</u>: $X \xrightarrow{b} \mathbb{P}^2$ blowup at p.

$L = b^* \mathcal{O}(3)$. L is nef (pullback of ample).

$K_X = b^* \mathcal{O}(-3) + E$. $\Rightarrow L + K_X \equiv E$

Take $D = E$. Then $E \cdot L = 0$ and $E^2 = -1$.

(Take $L = b^* \mathcal{O}(4)$, and the first condition of 2 is satisfied.)

Exercise: Find examples satisfying all of the other conditions in Reider's Theorem.

Cor: L ample on X. $L^2 \geq 5$ and $L \cdot C \geq 2$ for irreducible curves $C \subset X$. Then $\mathcal{O}(K_X + L)$ is globally generated.
If $L^2 \geq 10$ and $L \cdot C \geq 3$ $\forall$ $C \subset X$, then $\mathcal{O}(K_X + L)$ is very ample.

Cor (dim 2 of Fujita's Conjecture): A ample on X. Then $K_X + 3A$ is glob. gen. and $K_X + 4A$ is very ample.

Cor: (Kodaira '68, Bombieri '73) Let X be minimal of gen. type ($\Rightarrow K_X$ big + nef)

(i.) $\mathcal{O}(mK_X)$ is globally generated if $m \geq 4$ or if $m \geq 3$ and $K_X^2 \geq 2$

(ii.) $\Phi_{|mK_X|}$ is an embedding away from (-2)-curves if $m \geq 5$ or if $m \geq 3$ and $K_X^2 \geq 2$

Pf of (i.): Set $L = (m-1)K_X$. Then $L^2 = (m-1)^2 K_X^2 \geq 6$

Assume $|mK_X|$ has base points,

Case 1: $D \cdot L = 0$ and $D^2 = -1$.

Adjunction formula $\Rightarrow -1 = D^2 = D\cdot(D+K_X) \equiv 0 \pmod 2$
$\Rightarrow\Leftarrow$

Case 2: $D\cdot L = 1$ and $D^2 = 0$ contradicted by $m-1 \geq 2$

Pf of (ii): Similar.

More applications of Reider's Thm later.

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Proof of Reider's theorem:

(i) L a nef l.b. on X, $L^2 \geq 5$, s.t. $x \in X$ a base point of $\mathcal{O}(K_X+L)$.

$\Rightarrow \{x\}$ satisfies C-B w.r.t. $|K_X+L|$. So earlier theorem implies $\exists$ rk 2 v.b. E sitting in SES:

$$(*) \qquad 0 \to \mathcal{O}_X \to E \to L \otimes I_x \to 0$$

Recall: $c_1(E) = [L]$ and $c_2(E) = 1$

$$\Rightarrow c_1(E)^2 - 4c(E) = L^2 - 4 > 0.$$

So we can apply Bogomolov's Instability and get

$$(**) \qquad 0 \to A \to E \to B \otimes I_Z \to 0$$

where $Z \subset X$ is finite and $(A-B)^2 > 0$, $(A-B)\cdot H > 0$ for all ample H.

$\det(E) = L = A+B \Rightarrow B = L - A$

$\Rightarrow (2A-L)^2 > 0$ and $(2A-L)\cdot H > 0 \ \forall$ ample H.

$\Rightarrow 2A.H > L\cdot H \geq 0$ (a pos. mult. of H is effective)

$\Rightarrow A^* \cdot H < 0$

$\Rightarrow H^0(A^*) = 0 \Rightarrow \mathrm{Hom}(A, \mathcal{O}_X) = 0$

Denote by $\alpha : A \to L \otimes I_x$ the composition $A \hookrightarrow E \twoheadrightarrow L \otimes I_x$

$\alpha \neq 0$ since A doesn't map to $\mathcal{O}$.

So $H^0(\mathcal{O}(L-A) \otimes I_x) \neq 0 \Rightarrow$ There is some global section of $\mathcal{O}(L-A)$ vanishing along x

So $L - A = D$ effective, $x \in D \Rightarrow A = L - D$

<u>Collect inequalities</u>

L is a limit of ample divisors $\Rightarrow (2A-L)\cdot L = (L-2D)\cdot L \geq 0$

$L^2 \geq 5 \Rightarrow$ Hodge index implies that pairing is negative definite on orthogonal complement. Since $(L-2D)^2 > 0$,

$(L-2D)\cdot L \neq 0 \Rightarrow (L-2D)L > 0$ ①

<u>Exercise</u>: Use Hodge index to show that $(L^2)(D^2) \leq (L\cdot D)^2$ ②

Recall $1 = c_2(E) = A \cdot B + \text{length}(Z)$

and $A = L - D$, $B = L - A = D$

$\Rightarrow (L-D).D \leq 1$ ③

Finally, claim: $2D^2 < L \cdot D$ ④

If $D^2 > 0$, then $L \cdot D \neq 0$ by Hodge index.

$$\underset{①}{(L-2D) \cdot L > 0} \Rightarrow 2(L \cdot D) < L^2 \underset{②}{\leq} \frac{(L \cdot D)^2}{D^2}$$

$$\Rightarrow 2D^2 < L \cdot D$$

If $D^2 = 0$, then Hodge index $\Rightarrow L \cdot D \neq 0 \Rightarrow L \cdot D > 0$, so claim holds.

If $D^2 < 0$, $L \cdot D \geq 0$, so it holds.

$$L \cdot D - 1 \underset{③}{\leq} D^2 \underset{④}{<} \frac{L \cdot D}{2} \Rightarrow 0 \leq L \cdot D < 2$$

$\Rightarrow$ If $L \cdot D = 0$, $D^2 = -1$.

If $L \cdot D = 1$, $D^2 = 0$ □

Exercise: Prove the second part of Reider's Theorem.

Example: Let X be a smooth projective surface w/ $K_X = \mathcal{O}_X$ (i.e. X is K3 or abelian). L ample on X.

Claim: Assume $L^2 \geq 5$. L is not glob. generated $\iff$ X contains an irreducible curve $C \subseteq X$ w/ $p_a(C) = 1$ and $C \cdot L = 1$.

Pf: If L is not globally generated, then Reider's Theorem $\Rightarrow$ $\exists$ effective D s.t. $D^2 = -1$ or 0.

But adjunction $\Rightarrow$ D^2 is even so $D^2 = 0$ and $L \cdot D = 1$ $\Rightarrow$ $D = C$, irreducible, and $C^2 = 2g - 2 = 0 \Rightarrow g = 1$.

Assume L is globally generated, and $\exists$ such a C.

$L \cdot C = 1 \Rightarrow \phi_{|L|}(C)$ has degree 1. $\Rightarrow$ $C \cong \mathbb{P}^1$, which contradicts $p_a(C) = 1$. $\square$

Claim: If $L^2 \geq 10$ and L is b.p.f, then L is not very ample $\iff$ $\exists$ reduced curve D $p_a(D) = 1$ and $D \cdot L = 2$.

Pf: Assume L not v.a. then Reider's Thm $\Rightarrow$ exists D w/ either

a.) $D \cdot L = 0$, contradicts ampleness

b.) $D \cdot L = 1$ and $D^2 = 0$ or $\cancel{-1}$

In this case, by previous part, L is not b.p.f.

c.) $D \cdot L = 2$, $D^2 = 0 = 2p_a(D) - 2 \Rightarrow p_a(D) = 1$.
If D is non-reduced, $D \cdot L = 2 \Rightarrow D = 2C$.

But then $C \cdot L = 1$ and $C^2 = 0$, contradicting previous claim.

Assume now L is v. ample. Assume such a D exists.

Since $D \cdot L = 2$, D is irreducible or $D = C + C'$

If $D = C + C'$, then $\phi_{|L|}$ embeds each as a line.

But $p_a(D) = 1$, so C and C' intersect in exactly 2 points, which is a contradiction.

So D is irreducible and embeds as a degree 2 curve.
$\Rightarrow D \cong \mathbb{P}^1 \Rightarrow\Leftarrow$

Note: This easily generalizes to surfaces w/ K_X numerically trivial by replacing L w/ $K_X + L$.